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A quest for fundamental theorems of social choice

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Abstract We explore the frontier between possibility and impossibility results by analyzing different combinations of "pro-socialness" and "consistency" conditions. This exercise delivers stronger versions of four classical impossibility theorems, and offers a thorough understanding of connections among them. We also characterize social choice functions that are "independent of irrelevant alternatives," which makes evident that the fundamental difficulty of social choice lies in "pairwise" consistency requirements. We also introduce a concise pedagogical approach to classical impossibility theorems.

1 Introduction

Finding the minimal assumptions that are necessary for a proposition reveals the fundamental logic underlying it. By systematically relaxing assumptions employed by classical impossibility theorems in the field of social choice and welfare, we reach a family of theorems that each require an "existential pro-socialness condition" and a "pairwise consistency condition." We work with both "social choice functions (SCFs)" and "social preference functions (SPFs)."[1]

For SCFs, the list of consistency conditions includes monotonicity (Muller and Satterthwaite 1977), independence of irrelevant alternatives (IIA) (Denicolò 2000), and three newly formulated "Condorcetian" conditions that govern outcomes of "two-

[1] Kreps (2012, Chap. 8) suggests this terminology, which is more informative than "social welfare functions" (Arrow 1963) when we also work with social choice functions.

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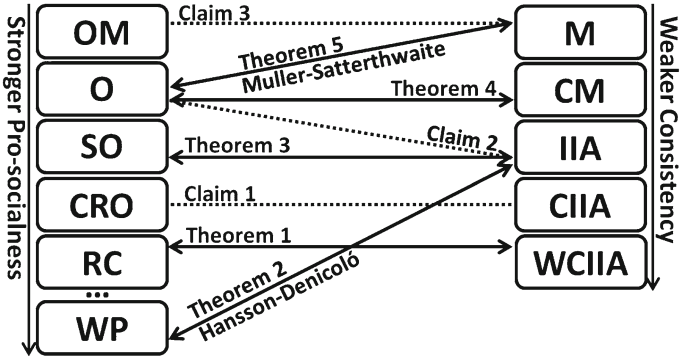


Fig. 1 Impossibility characterizations: *double arrows* and their counterclockwise rotations; possibility characterizations: *dotted lines* and their clockwise rotations

alternative runoffs." (A runoff preference profile has the same two alternatives on top of everyone's preference ranking.) The idea has been hinted at by several prominent researchers in the field,[2] but has never been stated formally. We can order those five conditions in terms of restrictiveness.

We have also explored a wide spectrum of pro-socialness conditions–for example, a Pareto efficiency requirement belongs to this category. By doing so, we are able to identify quite precisely the dividing lines between impossibility and possibility characterizations, i.e., whether the combination leads to dictatorship or not.

The results are summarized in Fig. 1. Some notable findings include the sufficiency of existential pro-socialness conditions, a complete characterization of IIA in Sect. 5, and the fact that combined with "onto," "Condorcet monotonicity," a condition strictly weaker than monotonicity, is enough for dictatorship (Theorem 4). Yu (2013) shows that both Arrow's theorem (Arrow 1963, Chap. 5) and the theorem of Gibbard (1973) and Satterthwaite (1975) are straightforward corollaries of Theorem 4. By contrast, it is known to be difficult to derive Arrow's theorem from a weaker theorem that employs onto and monotonicity. That the impossibility result remains under Condorcet monotonicity indicates that it is pairwise consistency that drives the impossibility theorem.

The main impossibility theorem in Sect. 4 shows how to relax those two conditions in the theorem of Hansson (1969) and Denicolò (1993). For example, "Condorcet's IIA" requires choices to be "generalized Condorcet winners," that is, they do not lose to any alternatives in two-alternative runoffs. The name comes from an analogy with Condorcet's paradox[3] (Condorcet 1785), where a Condorcet winner is required to win any two-alternative competitions. This characterization is less susceptible to the

[2] Gibbard (1973) proves his theorem by "bringing two alternatives to the top." Sen (2001) and Aswal et al. (2003) articulate the importance of top positions.

[3] When a society of 3 voters have preferences over 3 candidates as $a_1 \succ_1 a_2 \succ_1 a_3, a_2 \succ_2 a_3 \succ_2 a_1$, and $a_3 \succ_3 a_1 \succ_3 a_2$ respectively, none of the candidates has a convincing case of winning. For instance, a_1 would have lost to a_3 in a "two-candidate runoff" determined by simple majority, if a_2 dropped out of the race or become the least favored due to some devastating scandals. But a_3 would have lost to a_2, and a_2 to a_1, denying the existence of any "Condorcet winner.'

criticism that IIA is unreasonable (Kreps 1990, p. 177). The theorem is also immune to the criticism (Sen 1970, Chap. 3) that decision making does not necessarily require social preferences or rationality axioms such as Arrow's choice axiom (Arrow 1959). Thus, we can conclude that pairwise consistency conditions–even those weaker than IIA–can easily lead to difficulties in social choice.

The corresponding SPF version in Sect. 6 relaxes both conditions in Arrow's theorem simultaneously.[4] The discussion also leads to other SPF impossibility theorems closely related to SCF ones, confirming for SPFs the principle that impossibility theorems only need to assume pairs of existential pro-socialness conditions and pairwise consistency conditions. For brevity, we leave out other important impossibility characterizations, which could also be reformulated following our approach, such as, for instance, Grether and Plott (1982), Jackson and Srivastava (1996), and Dutta et al. (2001). The connections are well-explored by Man and Takayama (2013). Section 7 concludes, and the appendix discusses an alternative pedagogical approach to classical impossibility theorems, taking advantage of our short proof of the Hansson–Denicolò theorem.

2 Setup and conditions of "pro-socialness" and "consistency"

We consider a set of individuals $\mathcal{N} = \{1, \ldots, N\}$ with $N \geq 2$, each having some preferences over $M \geq 3$ alternatives $A = \{a_1, \ldots, a_M\}$. Throughout the paper, we label individuals with natural numbers $n, n' \in [1, N]$ and use $i, j, k, l, m \in [1, M]$ for alternatives. Let $\mathcal{P}$ be the set of all possible relations on M that are complete, asymmetric, and transitive, i.e., strict preferences. The set of *preference profiles* $\mathcal{P}^{\mathcal{N}}$ is a collection of functions from $\mathcal{N}$ to $\mathcal{P}$ that specify preferences of each individuals. In other words, an element of $\mathcal{P}^{\mathcal{N}}$ is an ordered list $\vec{\succ} = (\succ_1, \ldots, \succ_N)$ with $\succ_n \in \mathcal{P}$ for every $1 \leq n \leq N$.

A *social choice function (SCF)* $F : \mathcal{P}^{\mathcal{N}} \to A$ assigns to each $\vec{\succ} \in \mathcal{P}^{\mathcal{N}}$ a choice $F(\vec{\succ}) \in A$. A *social preference function (SPF)* $R : \mathcal{P}^{\mathcal{N}} \to \mathcal{P}$ assigns to each $\vec{\succ} \in \mathcal{P}^{\mathcal{N}}$ social preferences $R(\vec{\succ}) \in \mathcal{P}$.

We restrict the analysis to strict preferences for the sake of brevity, as the version with indifference[5] is sometimes lengthy. Moreover, strict preferences are usually generic when we consider processes that generate them.

Several intuitive definitions may facilitate the subsequent exposition. For profile $\vec{\succ}$, a_i *dominates* a_j if $a_i \succ_n a_j$ for every n. Profiles $\vec{\succ}$ and $\vec{\succ}'$ *agree on* $\{a_i, a_j\}$ if $a_i \succ_n a_j$ is necessary and sufficient for $a_i \succ'_n a_j$ for every n. Profile $\vec{\succ}'$ *maintains* a_i*'s positions*[6] in $\vec{\succ}$ if $a_i \succ_n a_j$ implies $a_i \succ'_n a_j$ for every n and every $j \neq i$.

[4] Denicolò (1998) relaxes AIIA to "relational independent decisiveness," inspired by "independent decisiveness" of Sen (1993) for social choice correspondences, but considers it difficult to find arguments in favor of it. Campbell and Kelly (2002) survey the topic.

[5] Available on my personal website: http://sites.google.com/site/neilningyu/. The readers can also consult Man and Takayama (2013) for interesting features when weak preferences are allowed–for example, serial dictatorship.

[6] Mas-Colell, Whinston, and Green (1995, p. 808)

$A' \subset A$ *tops* $\vec{\succ}$ or is *at the top* if $a_i \succ_n a_j$ for every n, every $a_i \in A'$, and every $a_j \in A \backslash A'$. If in addition $A' = \{a_i, a_j\}$ for some i, j, we call $\vec{\succ}$ a $\{i, j\}$-*runoff*[7] or $\{a_i, a_j\}$-*runoff*. A *topping function*[8] $T_{ij} : \mathcal{P}^{\mathcal{N}} \to \mathcal{P}^{\mathcal{N}}$ brings $\{a_i, a_j\}$ to the top keeping other rankings intact.[9] For fixed SCF F and SPF R, we define the *topping choice function* $F_{ij} \equiv F \circ T_{ij}$ and the *topping preference function* $R_{ij} \equiv R \circ T_{ij}$.

If we can identify a *social choice dictator*, then F is a *dictatorship*.

Definition 1 (*D*). A SCF F is *dictatorial* if there exists a *dictator* n such that $F(\vec{\succ}) \succ_n a_i$ for every $a_i \neq F(\vec{\succ})$ and every $\vec{\succ}$.

A dictatorship is certainly undesirable, but a recurring theme in the field of social choice is the impossibility of avoiding it if we impose reasonable assumptions on SCFs and SPFs.[10] If two assumptions (X) and (Y) on F imply that F is a dictatorship, we call the pair an *impossibility characterization*; otherwise, we have a *possibility characterization*. (D) implies all SCF conditions in this paper, so the converses of all our impossibility theorems are true. Fixing (X), we are interested in how weak (Y) could be for generating dictatorship. Our strategy is to categorize interpretable assumptions into two collections linearly ordered by restrictiveness, and comb through all combinations. The first collection features "pro-socialness."

Definition 2 (*WP*). A SCF F is *weakly Paretian* if a_i dominating a_j implies $F(\vec{\succ}) \neq a_j$ for every i, j and every $\vec{\succ}$.

Definition 3 (*RU*). A SCF F is *runoff unanimous* if $F(\vec{\succ}) = a_i$ for every runoff $\vec{\succ}$ with $\{a_i\}$ at the top.

Definition 4 (*RO*). A SCF F is *runoff onto* if for every i, j, $F(\vec{\succ}) = a_i$ for some $\{i, j\}$-runoff $\vec{\succ}$ with $\{a_i\}$ at the top.

Definition 5 (*RC*). A SCF F is *runoff connected*[11] if for every i, j, there exits a sequence of alternative$\{a_{(0)}, \ldots, a_{(K)}\}$ such that $a_{(0)} = a_i$, $a_{(K)} = a_j$, and for each $k = 1, \ldots, K$, $F(\vec{\succ}^k) = a_{(k-1)}$ for some $\{a_{(k-1)}, a_{(k)}\}$-runoff $\vec{\succ}^k$ with $\{a_{(k-1)}\}$ at the top.

Definition 6 (*CRO*). A SCF F is *circle runoff onto*[12] if for every i, (1) $F(\vec{\succ}) = a_i$ for some runoff $\vec{\succ}$ with $\{a_i\}$ at the top, and (2) $F(\vec{\succ}') = a_j$ for some $j \neq i$ and some $\{i, j\}$-runoff $\vec{\succ}'$ with $\{a_j\}$ at the top.

[7] If F is a two-round system which automatically singles out two candidates first, it is desirable that A' enter the second round.

[8] Gibbard (1973) employs this function in his proof.

[9] Mathematically, when $\vec{\succ}' = T_{ij}(\vec{\succ})$, for every n, first, $a_i \succ_n a_j$ if and only if $a_i \succ'_n a_j$; second, $a_i \succ'_n a_k$ and $a_j \succ'_n a_k$ for every $k \notin \{i, j\}$; third, $a_l \succ_n a_m$ if and only if $a_l \succ'_n a_m$ for every $l, m \notin \{i, j\}$.

[10] SPF dictatorship remains undefined until the section devoted to SPFs.

[11] A referee suggested this condition.

[12] (CRO) can be satisfied with the existence of multiple circles of alternatives winning runoffs in a circular manner.

Definition 7 (*SO*). A SCF F is *subtly onto* if $F(\mathcal{P}^{\mathcal{N}}) = A$, and there exist i, j, and $\vec{\succ}$ such that $F(\vec{\succ}) = a_i$, where a_i dominates a_j

Definition 8 (*O*). A SCF F is *onto* if $F(\mathcal{P}^{\mathcal{N}}) = A$.

Definition 9 (*OM*). A SCF F is *onto minus* if $|F(\mathcal{P}^{\mathcal{N}})| \geq M - 1$.

(WP) and (RU) are *universal* pro-socialness conditions with "for all" quantifying preference profiles, while others are *existential* claims. Many theorems take (WP) for granted, but there exist arguments against it (Sen 1970, Chap. 11). Unfortunately, more undisputable existential claims are always restrictive enough for impossibility results in our investigations. Although we have explored a much bigger collection, we have arrived at three close pairs, (RC)-(CRO), (SO)-(O), and (O)-(OM), which form dividing lines between impossibility and possibility characterizations. It is easy to confirm that conditions 2 to 9 are in decreasing order of restrictiveness.

Another collection of conditions demand "consistency" in social choices.

Definition 10 (*M*). A SCF F is *monotonic* if whenever $F(\vec{\succ}) = a_i$ and $\vec{\succ}'$ maintains a_i's positions in $\vec{\succ}$, $F(\vec{\succ}') = a_i$.

Note that (M) is not a *pairwise* condition. By contrast, the other conditions effectively focus on the consistency of two-alternative choices.

Definition 11 (*CM*). A SCF F is *Condorcet monotonic* if whenever $\vec{\succ}$ and $\{i, j\}$-runoff $\vec{\succ}'$ agree on $\{a_i, a_j\}$, $F(\vec{\succ}) = a_i$ implies $F(\vec{\succ}') = a_i$.

(CM) is sufficient but not necessary for requiring that $F_{ij}(\vec{\succ}) = a_i$ whenever $F(\vec{\succ}) = a_i$, a weaker condition that plays a part in the proof of Gibbard (1973).

Definition 12 (*IIA*). A SCF F is *independent of irrelevant alternatives*[13] if whenever $\vec{\succ}$ and $\vec{\succ}'$ agree on $\{a_i, a_j\}$, $F(\vec{\succ}) = a_i$ implies $F(\vec{\succ}') \neq a_j$.

Definition 13 (*CIIA*). A SCF F is *Condorcet independent of irrelevant alternatives* if whenever $\vec{\succ}$ and $\{i, j\}$-runoff $\vec{\succ}'$ agree on $\{a_i, a_j\}$, $F(\vec{\succ}) = a_i$ implies $F(\vec{\succ}') \neq a_j$.

Definition 14 (*WCIIA*). A SCF F is *weakly Condorcet independent of irrelevant alternatives* if whenever $\vec{\succ}$ and $\{i, j\}$-runoff $\vec{\succ}'$ agree on $\{a_i, a_j\}$, where $\vec{\succ}$ is not a $\{i, j\}$-runoff, $F(\vec{\succ}) = a_i$ implies $F(\vec{\succ}') \neq a_j$.

(CIIA) requires any social choice to be a *generalized Condorcet winner* in that it does not lose to an opponent in a two-candidate runoff that keeps their relative rankings intact. Otherwise, it would be difficult to justify that $F(\vec{\succ}) = a_i$ to those who favor a_j over a_i, because the counterfactual of two-candidate competition shows that a_j is superior. (IIA) concerns people making a case out of any other profiles, not only salient profiles like runoffs. (CM) links to a stronger generalization of the Condorcet winner concept: a winner sweeps any two-candidate runoff it enters as long as its relative rankings against the runoff opponent are intact. All consistency conditions may be regarded as trying to eliminate "complaints" about internal inconsistency of social choices.

[13] This is similar to "Hansson's pairwise independence" named by Denicolò (2000), who explains that R. Cagliozzi shows its equivalence to "Hansson's independence" in Hansson (1969).

Lemma 1 *Conditions 10 to 14 are in decreasing order of restrictiveness.*

Proof (CM) regulates only "maintenances" that result in runoffs, so (M) implies (CM). To see that (M) is not necessary, let $N \geq 3$ and $F(\mathcal{P}^{\mathcal{N}}) = \{a_1, a_2\}$. It picks between $\{a_1, a_2\}$ the one that a *restricted anti-dictator* n dislikes unless everyone puts the same alternative above the other. To confirm (CM), we reason that a_1 is selected either as the dominant alternative or as an undominated one that is worse for n. In the first case, entering a corresponding runoff keeps a_1 dominant. In the second case, a runoff against a_2 keeps their relative status, but a runoff against $a_i \notin \{a_1, a_2\}$ makes a_1 dominant. The case for a_2 is symmetric. When others have different opinions, simply moving a_1 up for n could result in its elimination, a violation of (M).

To show (IIA), let F satisfy (CM) and $\vec{\succ}$ and $\vec{\succ}'$ agree on $\{a_i, a_j\}$. Suppose $F(\vec{\succ}) = a_i$ and $F(\vec{\succ}') = a_j$. By (CM), $F_{ij}(\vec{\succ})$ should be both a_i and a_j, a contradiction. An *anti-dictatorship* violates (CM) but satisfies (IIA), that is, F always selects the least favorite alternative of a fixed individual n.

(IIA) clearly implies (CIIA). To show that the converse does not hold, we let $M = 3$ and construct a "bone assignment" F as follows.[14] Whenever a_i and a_j enter a runoff, a third alternative a_k wins, but F always picks a_1 otherwise. The conclusion in (CIIA) is always true. When $a_1 \succ_n a_2 \succ_n a_3$ for some n and $a_1 \succ_{n'} a_3 \succ_{n'} a_2$ for the others, the choice is a_1. When $a_1 \succ'_n a_2 \succ'_n a_3$ for every n, the choice is a_3. But these two profiles agree on $\{a_1, a_3\}$, violating (IIA).

(CIIA) implies (WCIIA), for (WCIIA) imposes restrictions only on more generic profiles. Let $M = 5$. Consider a *restricted anti-dictator* n' whose least favorite among $\{a_1, a_2, a_3\}$ is always the social choice, except that $F(\vec{\succ}'') = a_4$ when $a_4 \succ''_n a_5 \succ''_n a_1 \succ''_n a_2 \succ''_n a_3$ for every n and that $F(\vec{\succ}''') = a_5$ when $a_4 \succ'''_n a_5 \succ'''_n a_3 \succ'''_n a_2 \succ'''_n a_1$ for every n. The violation of (CIIA) comes from two exceptions $\vec{\succ}''$ and $\vec{\succ}'''$. To see (WCIIA), we analyze two cases. For the case of $\{i, j\} = \{4, 5\}$, $F(\vec{\succ}) \notin \{a_i, a_j\}$ when $\vec{\succ}$ is not a $\{i, j\}$-runoff. For the other cases, $F(\vec{\succ}') \notin \{a_i, a_j\}$ when $\vec{\succ}'$ is a $\{i, j\}$-runoff. □

3 Preview of results

Figure 1 previews our findings. Any pair of conditions connected by a double arrow leads to dictatorship, but we can construct non-dictatorial SCFs for any pair with dotted lines. From top to bottom, pro-socialness conditions strengthen and consistency conditions weaken. Consequently, each counterclockwise rotation of a double arrow around a center on it can only give rise to an impossibility characterization, and each clockwise rotation of a dotted line around a center on it can only generate a possibility characterization.

Assuming (CIIA), there must exist a weakest condition (X) for dictatorship that lies between (RC) and (CRO), making sure that any weaker pro-socialness condition admits counterexamples. For (IIA), the leap happens between (SO) and (O). (O) seems to be the absolute minimum in that (OM), barely different from (O), admits non-dictatorial SCFs when combined with the strongest consistency condition (M).

[14] This F honors the proverb "two dogs strive for a bone and the third one runs off with it."

4 A "Condorcetian" impossibility theorem

(CIIA) is analogous to the idea behind "Condorcet winner," and (WCIIA) restricts the requirement to more generic profiles. (RC) is reminiscent of the circular structure in Condorcet profiles.

Theorem 1 *If a SCF F satisfies (RC) and (WCIIA), then it is dictatorial*

Proof Lemmas below show that (RC) and (WCIIA) together necessitate (WP) and (IIA), which imply (D) by the Hansson–Denicolò theorem, proven succinctly afterwards. □

Lemma 2 *(RC) and (WCIIA) imply (RO).*

Proof Fix $a \in A$. For each $b \neq a$, fix a sequence of shortest length among those fulfilling (RU) requirements for (a, b). Define $d_a(b)$ to be this length and $d_a(a) = 0$. Define $p(b)$ be the alternative immediately before b in this sequence. Notice that $d_a(p(b)) < d_a(b)$.

Now fix $b \neq a$. If $p(b) = a$, the condition for (RO) regarding a and b is satisfied. Otherwise, construct a preference profile $\vec{\succ}$ such that for each individual n, (1) $a \succ_n b \succ_n c$ for all $c \neq a, b$, and (2) for each $c, c' \in A\backslash\{a, b\}$, $c \succ_n c'$ if $d_a(c) < d_a(c')$. We show $F(\vec{\succ}) = a$ by ruling out other alternatives.

To see $F(\vec{\succ}) \neq b$, take an alternative $c \neq b$ such that $p(c) = a$, which certainly exists as there must be an alternative immediately after a in the sequence from a to b. Since a wins a $\{a, c\}$-runoff $\vec{\succ}^{a,c}$ with $\{a\}$ on top, and $\vec{\succ}^{a,c}$ agrees with $\vec{\succ}$ on $\{a, b\}$, by (WCIIA), $F(\vec{\succ}) \neq b$.

To see $F(\vec{\succ}) \neq c$ for all $c \neq a, b$, recall that $p(c)$ wins a $\{p(c), c\}$-runoff $\vec{\succ}^{p(c),c}$ with $p(c)$ on top. Since $d_a(p(c)) < d_a(c)$, $\vec{\succ}$ and $\vec{\succ}^{p(c),c}$ agree on $\{p(c), c\}$. Suppose $F(\vec{\succ}) = c$. Then (WCIIA) requires $F(\vec{\succ}^{p(c),c}) \neq p(c)$, a contradiction. □

Lemma 3 *(RO) and (WCIIA) imply (RU).*

Proof Suppose $F(\vec{\succ}) \neq a_i$ for a $\{i, j\}$-runoff $\vec{\succ}$ with $\{a_i\}$ at the top. In one case, $F(\vec{\succ}) = a_k$ for $k \notin \{i, j\}$, implying by (WCIIA) that $F(\vec{\succ}') \neq a_i$ for any $\{i, k\}$-runoff $\vec{\succ}'$ with $\{a_i\}$ at the top, a contradiction to (RO). In the other case, $F(\vec{\succ}) = a_j$. Consider a non-runoff $\vec{\succ}''$ with $a_i \succ''_1 a_k \succ''_1 a_j$ and $a_i \succ''_n a_j \succ''_n a_k$ for every $n \neq 1$ and every $k \notin \{i, j\}$. By (WCIIA), $F(\vec{\succ}'') = a_i$ is inconsistent with $F(\vec{\succ}) = a_j$; $F(\vec{\succ}'') = a_j$ is inconsistent with $F(\vec{\succ}''') = a_i$ for some $\{i, j\}$-runoff $\vec{\succ}'''$ with $\{a_i\}$ at the top; $F(\vec{\succ}'') = a_k$ is inconsistent with $F(\vec{\succ}'''') = a_i$ for some $\{i, k\}$-runoff $\vec{\succ}''''$ with $\{a_i\}$ at the top. □

Lemma 4 *(RU) and (WCIIA) imply (WP).*

Proof Let $a_i \succ_n a_j$ for every n. If $\vec{\succ}$ is a $\{i, j\}$-runoff, $F(\vec{\succ}) = a_i$ by (RU). If $\vec{\succ}$ is not a $\{i, j\}$-runoff, suppose $F(\vec{\succ}) = a_j$. By (WCIIA), $F_{ij}(\vec{\succ}) \neq a_i$, but either choice contradicts (RU). □

Lemma 5 *(WP) and (WCIIA) imply (CIIA).*

Proof We only need to establish that whenever $\{i, j\}$-runoffs $\vec{\succ}$ and $\vec{\succ}'$ agree on $\{a_i, a_j\}$, $F(\vec{\succ}) = a_i$ implies $F(\vec{\succ}') \neq a_j$. Suppose $F(\vec{\succ}') = a_j$. Perturb $\vec{\succ}$ by moving a_k to be between a_i and a_j for individual 1. The new profile $\vec{\succ}''$ is not a runoff, but agrees with both $\vec{\succ}$ and $\vec{\succ}'$ on $\{a_i, a_j\}$. (WP) mandates $F(\vec{\succ}'') \in \{a_i, a_j\}$, which contradicts (WCIIA). □

Lemma 6 *(WP) and (CIIA) imply (IIA).*

Proof Let $\vec{\succ}$ and $\vec{\succ}'$ agree on $\{a_i, a_j\}$. If $F(\vec{\succ}) = a_i$ and $F(\vec{\succ}') = a_j$, $F_{ij}(\vec{\succ})$ is neither a_j nor a_i by (CIIA), violating (WP). □

Theorem 2 (Hansson–Denicolò). *If a SCF F satisfies (WP) and (IIA), then it is dictatorial.*

Proof [15] Pick any $\vec{\succ}'$ with $\{a_i\}$ and $\{a_i, a_j\}$ at the top. Swap the positions of $\{a_i, a_j\}$ sequentially from 1 to N. By (WP), the choice is either a_i or a_j, starting with a_i and ending with a_j. The (i, j)-*pivotal voter* n_{ij} is the first whose swap makes a difference. By (IIA), the definition is independent of which $\vec{\succ}'$ (with $\{a_i\}$ and $\{a_i, a_j\}$ at the top) we start with.

Consider arbitrary $\vec{\succ}''$ and $\vec{\succ}'''$ with the depicted rankings of $\{a_i, a_j, a_k\}$ at the top, where squares mark possible positions of a_k. The definition of n_{ij} requires $F_{ij}(\vec{\succ}'') = a_i$ and $F_{ij}(\vec{\succ}''') = a_j$, so $F(\vec{\succ}'') \neq a_j$ and $F(\vec{\succ}'') \neq a_i$ by (IIA). We get $F(\vec{\succ}'') = a_i$, for (WP) rules out a_k and others. As $\vec{\succ}''$ and $\vec{\succ}'''$ agree on $\{a_i, a_k\}$, $F(\vec{\succ}''') \neq a_k$ by (IIA) and we are left with $F(\vec{\succ}''') = a_j$. Focusing on a_j and a_k in $\vec{\succ}'''$, we conclude by (IIA) that

$$a_j \succ_{n_{ij}} a_k \text{ implies } F(\vec{\succ}) \neq a_k. \qquad (*)$$

	1	$\ldots$	$n_{ij}-1$	n_{ij}	$n_{ij}+1$	$\ldots$	N
$\vec{\succ}''$	a_j	$\ldots$	a_j	a_i	a_i	$\ldots$	a_i
	a_k	$\ldots$	a_k				
	a_i	$\ldots$	a_i	a_j	a_j	$\ldots$	a_j
				a_k	a_k	$\ldots$	a_k
$\vec{\succ}'''$	□	$\ldots$	□				
	a_j	$\ldots$	a_j	a_j	a_i	$\ldots$	a_i
	□	$\ldots$	□		□	$\ldots$	□
	a_i	$\ldots$	a_i	a_i	a_j	$\ldots$	a_j
				a_k	□	$\ldots$	□

In the swapping process that defines n_{jk}, $(*)$ says that the choice remains a_j as long as n_{ij} ranks j above k, so $n_{jk} \geq n_{ij}$. For n_{kj}, the choice should become a_j no later than n_{ij} makes the change, so $n_{kj} \leq n_{ij}$. We have $n_{jk} \geq n_{ij} \geq n_{kj}$. As j and k are arbitrary, $n_{kj} \geq n_{jk}$ also holds, implying $n_{jk} = n_{kj} = n_{ij}$, which can be easily extended to any n_{lm}. But $(*)$ requires this unique pivotal voter to be a dictator. □

[15] The proof adapts Yu (2012), who in turn builds on Barberà (1980), Geanakoplos (2005), and Reny (2001) who developed and perfected the "pivotal voter" technique.

Once we replace (RC) with a slightly weaker condition (CRO), a possibility result emerges.

Claim 1 There exists a non-dictatorial SCF F that satisfies (CRO) and (CIIA).

Proof Consider a *disgruntled dictator* n when $M = 6$. Form three circles $A_1 = \{a_1, a_2\}, A_2 = \{a_3, a_4\}$, and $A_3 = \{a_5, a_6\}$. With $a_k \succ_n a_l \succ_n a_m$ for every $m \notin \{k, l\}$, one non-dictatorial SCF distinguishes two cases. When $\{a_k, a_l\}$ are in the same circle, the choice is a_k; when $\{a_k, a_l\}$ come from two different circles, the choice is the preferred alternative of n in the third circle. To see (CIIA), note that when $\{a_i, a_j\}$ are in different circles, they are never chosen in any $\{i, j\}$-runoff, so a violation is impossible. When $\{a_i, a_j\}$ are in the same cycle, n's preferred one, say a_i, is chosen in any $\{i, j\}$-runoff, but a_j is never chosen in any case with $a_i \succ_n a_j$, so a violation is still impossible. □

5 Other social choice impossibility theorems

Paired with (IIA), the barely sufficient pro-socialness condition for dictatorship is (SO).

Theorem 3 *If a SCF F satisfies (SO) and (IIA), then it is dictatorial.*

Proof It is a consequence of Theorem 2 and Lemma 7. □

Lemma 7 *(SO) and (IIA) imply (WP).*

Proof Let i, j, and $\vec{\succ}$ be those stated in (SO), by which we also have $F(\vec{\succ}') = a_j$ for some $\vec{\succ}'$. Generate $\vec{\succ}''$ by bringing $\{a_i\}$ to the top of $\vec{\succ}'$ without changing other rankings. (IIA) says that $F(\vec{\succ}'') \neq a_j$ due to $F(\vec{\succ}) = a_i$ and that $F(\vec{\succ}'') \neq a_k$ for $k \notin \{i, j\}$ due to $F(\vec{\succ}') = a_j$, so $F(\vec{\succ}'') = a_i$, which in turn implies by (IIA) that for every $k \neq i$, the choice is not a_k whenever a_i dominates a_k, that is, a_i has *shutout power*.

Let A' be the set of alternatives with shutout power. Suppose that there exists $a_l \notin A'$. Consider a profile with $F(\vec{\succ}''') = a_l$. We have $F_{il}(\vec{\succ}''') \neq a_i$ by (IIA) and $F_{il}(\vec{\succ}''') \neq a_k$ for $k \notin \{i, l\}$ thanks to the shutout power of a_i, so $F_{il}(\vec{\succ}''') = a_l$. Because a_l dominates some a_k for $k \notin \{i, l\}$ in $T_{il}(\vec{\succ}''')$, the proof in the previous paragraph applies with a_l taking the role of a_i, granting shutout power to a_l, a contradiction. We must have $A' = A$. □

(O) and (IIA) constitute a possibility characterization, but non-dictatorial SCFs we allow in are in a sense torturing, and thus more troublesome.

Definition 15 (*AntiD*). A SCF F is *anti-dictatorial* if there exists an *anti-dictator* n such that $a_i \succ_n F(\vec{\succ})$ for every $a_i \neq F(\vec{\succ})$ and every $\vec{\succ}$.

Claim 2 The only nondictatorial SCFs that satisfy (O) and (IIA) are anti-dictatorial.[16]

[16] The statement bears a similar structure to a SPF theorem of Wilson (1972).

Proof We first define a *preference reversing function* $\phi : \mathcal{P}^{\mathcal{N}} \to \mathcal{P}^{\mathcal{N}}$ with $\vec{\succ}' = \phi(\vec{\succ})$ saying that $a_i \succ'_n a_j$ if and only if $a_j \succ_n a_i$ for every i, j and every n. Let F satisfies (O) and (IIA) but not (D). By Theorem 3, F violates the second part of (SO), so SCF $F \circ \phi$ satisfies (WP). Since (IIA) clearly holds for $F \circ \phi$, it is dictatorial by Theorem 2, so F is anti-dictatorial. □

Theorem 4 suggests another Condorcetian impossibility characterization.

Theorem 4 *If a SCF F satisfies (O) and (CM), then it is dictatorial.*

Proof We have Theorem 3, Lemma 8, and sufficiency of (CM) for (IIA). □

Lemma 8 *(O) and (CM) imply (SO).*

Proof With $F(\vec{\succ}) = a_i$, a_i wins $T_{ij}(\vec{\succ})$, dominating a third a_k. □

Evidently, (CM) absorbs the minuscule amount of pro-socialness–better known as positive responsiveness–in (SO) so as to compensate for a neutral condition (O).

A natural question is whether we can work with (WCM) defined below, as we relax (CIIA) to (WCIIA). Lemma 9 shows that it is unnecessary.

Definition 16 (*WCM*). A SCF F is *weakly Condorcet monotonic* if whenever $\vec{\succ}$ and $\{i, j\}$-runoff $\vec{\succ}'$ agree on $\{a_i, a_j\}$, where $\vec{\succ}$ is not a $\{i, j\}$-runoff, $F(\vec{\succ}) = a_i$ implies $F(\vec{\succ}') = a_i$.

Lemma 9 *(WCM) is equivalent to (CM).*

Proof We only need to show that (WCM) implies (CM). Let $\vec{\succ}$ and $\{i, j\}$-runoff $\vec{\succ}'$ agree on $\{a_i, a_j\}$ and $F(\vec{\succ}) = a_i$. (WCM) leaves no worry unless $\vec{\succ}$ is a $\{i, j\}$-runoff. Suppose that $F(\vec{\succ}') = a_k$ for $k \neq i$. If $k \neq j$, we have $F_{ik}(\vec{\succ}) = a_i$ and a_k by (WCM), a contradiction. If $k = j$, modify $\vec{\succ}$ to get non-runoff $\vec{\succ}''$ by moving a third a_l between a_i and a_j for individual 1. Let $a_i \succ''_1 a_l \succ''_1 a_j$ without loss of generality. (WCM) says that $F(\vec{\succ}'') \notin \{a_i, a_j\}$, so $F(\vec{\succ}'') = a_m$ for $m \notin \{i, j\}$. By (WCM), $F_{im}(\vec{\succ}'')$ should be both a_i and a_m, a contradiction. □

Theorem 5 (Muller–Satterthwaite). *If F satisfies (O) and (M), then it is dictatorial.*

Proof (M) implies (CM). □

We can now investigate non-dictatorial SCFs that satisfy (OM) and (M), and show that by discriminating against one alternative, they are even less desirable than dictatorial ones.

Definition 17 (*DiscD*). A SCF F is *discriminatorily dictatorial* if there exist i and n such that for any $\vec{\succ}$, $F(\vec{\succ}) \in A \backslash \{a_i\}$ and $F(\vec{\succ}) \succ_n a_j$ for every $a_j \notin \{a_i, F(\vec{\succ})\}$.[17]

[17] Had we defined dictatorship as the existence of an individual whose favorite in the range of the social choice function is always chosen, a discriminatory dictatorship would be a dictatorship. The purpose of the following exposition is to rigorously demonstrate that the central difficulty of social choice lies in pairwise consistency conditions. Barberá (1983) defines dictatorship as the existence of an individual whose favorite in the range of the SCF is always chosen–in which case a discriminatory dictatorship would always be a dictatorship. Claim 3 then follows from Barberà's reformulation of the Gibbard–Satterwaite theorem (Barberá 1983, p. 415), since when $M > 3$, by (OM), the range of the SCF contains $M - 1 > 2$ elements.

Claim 3 The only nondictatorial SCFs that satisfy (OM) and (M) are discriminatorily dictatorial given $M > 3$.

Proof Consider the case of $|F(\mathcal{P}^{\mathcal{N}})| = M - 1$ with $M > 3$. Suppose that F does not satisfy (DiscD). For alternatives in $F(\mathcal{P}^{\mathcal{N}})$, F is not dictatorial. We can accordingly construct a non-dictatorial SCF that satisfies (O) and (M) for the case of $M - 1$ alternatives, violating Theorem 5. (We leave the straightforward construction to the reader.) □

Synthesizing claims 2 and 3, we note that a SCF F that satisfies (IIA) either imposes the restriction of $|F(\vec{\succ})| \leq 2$, or features dictatorship or anti-dictatorship for those alternatives it does consider.

Claim 4 If a SCF F satisfies (IIA), there are 4 mutually exclusive possibility:

1. $|F(\mathcal{P}^{\mathcal{N}})| > 2$ and F is dictatorial in choosing from $F(\mathcal{P}^{\mathcal{N}})$;
2. $|F(\mathcal{P}^{\mathcal{N}})| > 2$ and F is anti-dictatorial in choosing from $F(\mathcal{P}^{\mathcal{N}})$;
3. $|F(\mathcal{P}^{\mathcal{N}})| = 2$ and profiles agreeing on $F(\mathcal{P}^{\mathcal{N}})$ are assigned the same choice;
4. $|F(\mathcal{P}^{\mathcal{N}})| = 1$.

Namely, we have attained full knowledge of what constitutes the set of SCFs that satisfy (IIA), and formalized the claim of Samuelson (1985) that (IIA) almost always fail to hold for well-behaved choice methods. For analogous statements about SPFs, the readers can consult Campbell (1992a, b) and Campbell and Kelly (1993).

6 Social preference impossibility theorem

We can carry out a similar program for SPFs, but for conciseness, we focus on establishing the principle that a combination of an existential pro-socialness condition and a pairwise consistency condition is always sufficient for dictatorship. In the meantime, both conditions in Arrow's theorem are relaxed.

To understand the connection between SCFs and SPFs, we can for instance imagine a world with $M \geq 3$ songs produced per year. A firm collects personal preferences of $N \geq 2$ critics so as to publish an "annual ranking" and to present a "song of the year" award.[18] In devising an award algorithm or SCF, the firm can never succeed in preventing one critic's opinion from dominating and at the same time meeting any above-mentioned impossibility characterizations, which all seem reasonable. Arrow (1950) casts doubt on the possibility of a desirable ranking algorithm or SPF. Our derivations utilize that a ranking algorithm contains an award algorithm: the song of the year is often the number one song in ranking.

A SCF F^R is *derived from* a SPF R if for any $\vec{\succ}$, $a_i R(\vec{\succ}) a_j$ for every $j \neq i$ is necessary and sufficient for $F^R(\vec{\succ}) = a_i$. Notice that we cannot infer a SPF R^F from a SCF F, because F does not impose rankings between losing alternatives. In order words, asking for a SPF tends to be more restrictive than asking for a SCF. We can define (SP-X) conditions for R by requiring F^R to satisfy (X) conditions, but not the

[18] Arrow and Raynaud (1986) apply social choice theory to expert systems.

other way around. As an example, we spell out (SP-O), but leave similar definitions to the reader.

Definition 18 (*SP-O*). A SPF R is *onto* if F^R satisfies (O), i.e., for any i, there exists $\vec{\succ}$ such that $a_i R(\vec{\succ}) a_j$ for every $j \neq i$.

We attach "Arrow" to concepts undefinable in terms of F^R.

Definition 19 (*SP-AD*). A SPF R is *Arrow dictatorial* if there exists a *social preference dictator* n such that $a_i \succ_n a_j$ implies $a_i R(\vec{\succ}) a_j$ for every i, j and every $\vec{\succ}$.

Definition 20 (*SP-AU*). A SPF R is *Arrow unanimous* if $a_i \succ_n a_j$ for every n implies $a_i R(\vec{\succ}) a_j$ for every i, j and every $\vec{\succ}$.

Definition 21 (*SP-AM*). A SPF R is *Arrow monotonic* if whenever $a_i R(\vec{\succ}) a_j$ and $\vec{\succ}'$ maintains a_i's positions in $\vec{\succ}$, $a_i R(\vec{\succ}') a_j$.

Definition 22 (*SP-ACM*). A SPF R is *Arrow-Condorcet monotonic* if whenever $\vec{\succ}$ and $\{i, j\}$-runoff $\vec{\succ}'$ agree on $\{a_i, a_j\}$, $a_i R(\vec{\succ}) a_j$ implies $F^R(\vec{\succ}') = a_i$.

Definition 23 (*SP-AIIA*). A SPF R is *Arrow independent of irrelevant alternatives* if whenever $\vec{\succ}$ and $\vec{\succ}'$ agree on $\{a_i, a_j\}$, $a_i R(\vec{\succ}) a_j$ implies $a_i R(\vec{\succ}') a_j$.

Definition 24 (*SP-ACIIA*). A SPF R is *Arrow-Condorcet independent of irrelevant alternatives* if whenever $\vec{\succ}$ and $\{i, j\}$-runoff $\vec{\succ}'$ agree on $\{a_i, a_j\}$, $a_i R(\vec{\succ}) a_j$ implies $F^R(\vec{\succ}') \neq a_j$.

These concepts are stronger than (SP-D), (SP-WP), (SP-M), (SP-CM), (SP-IIA), and (SP-CIIA) respectively in that they also regulate losing alternatives. Specifically, that (SP-AIIA) implies (SP-IIA) offers us much information about the former, because Sect. 5 identifies every F that satisfies (IIA), none of which seems attractive. In retrospect, (SP-AIIA) is indeed a very strong assumption.

Resembling the interpretations of (CIIA) and (IIA), (SP-ACIIA) requires social preferences not to be contradicted by social choices in runoffs, avoiding protests similar to those in Condorcet's paradox, while (SP-AIIA) is more demanding, asking all profiles that agree on $\{a_i, a_j\}$ to bear the same social ranking over them. The bone assignment algorithm in the proof of Lemma 1 can be modified to bolster the claim that (SP-AIIA) does not imply (SP-ACIIA).

We list SPF impossibility theorems that are close analogues of Theorems 1–5. Interestingly, we fail to identify one for the Gibbard–Satterthwaite theorem (stated in the Appendix). Even for restrictive assumptions about individual preferences over social rankings, it is difficult, if not impossible, to deal with the multi-dimensional nature of social preferences. For the same reason, it is a bit harder to insist on consistency of social preferences than it is for choices.

Theorem 6 *If a SPF R satisfies (SP-RC) and (SP-ACIIA), then it is Arrow dictatorial.*

Proof As F^R satisfies (RC) and (CIIA), Theorem 1 says that F^R presents a social choice dictator n. She is a social preference dictator too. If $a_i \succ_n a_j$, individual n rules $F^R_{ij}(\vec{\succ}) = a_i$, so $a_i R(\vec{\succ}) a_j$ by (SP-ACIIA). □

Theorem 7 (Arrow's). *If a SPF R satisfies (SP-AU) and (SP-AIIA), then it is Arrow dictatorial.*

Proof (SP-AU) and (SP-AIIA) are respectively stronger than (SP-RC) and (SP-ACIIA). □

Theorem 8 *If a SPF R satisfies (SP-SO) and (SP-AIIA), then it is Arrow dictatorial.*

Proof As F^R satisfies (SO) and (IIA), it is dictatorial by Theorem 3, so R satisfies (SP-RC). To apply Theorem 6, we only need (SP-ACIIA), a weaker condition than (SP-AIIA). □

Theorem 9 *If a SPF R satisfies (SP-O) and (SP-ACM), then it is Arrow dictatorial.*

Proof As F^R satisfies (O) and (CM), it is dictatorial by Theorem 4, so R satisfies (SP-RC). To apply Theorem 6, we only need (SP-ACIIA), a weaker condition than (SP-ACM). □

Theorem 10 (Muller–Satterthwaite for SPFs). *If a SPF R satisfies (SP-O) and (SP-AM), then it is Arrow dictatorial.*

Proof (SP-AM) is stronger than (SP-ACM), and we have Theorem 9. □

7 Discussion

By combing through conceivable combinations of pro-socialness and consistency conditions, we obtain a novel map (Fig. 1) of possibility and impossibility characterizations for SCFs, and prove SPF impossibility theorems as close analogues. Coupled with any consistency condition, an existential pro-socialness condition is always sufficient for dictatorship, i.e., it is unnecessary to assume universal conditions such as Pareto efficiency (WP) for SCFs and unanimity (SP-AU) for SPFs. The unified exposition sheds light on the fact that the field of social choice embraces a coherent theme.

This exercise also pays homage to Condorcet: rationales behind several pairwise consistency conditions are analogous to those behind Condorcet's winner. The ideas underlying these conditions are essential, suggested by several pioneers in the field. Our formulation allows us to rigorously discuss their relations and entailments. For example, if F satisfies the weakest Condorcetian consistency condition (WCIIA), it either admits a dictator or violates (RC), a much weaker condition than (WP). With such strong theorems, we have shown that many criticisms fall not on substance, but on ancillary conditions such as social preferences and slightly mysterious "IIA" conditions. In practice, it becomes clear that designers of social choice mechanisms should not insist on consistency as long as they are minimally pro-social and against dictatorship.

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Appendix: An alternative proof of Arrow's theorem and the Gibbard–Satterthwaite theorem

While some of the proofs below are well-known, we include them in order to demonstrate an alternative approach of proving classical impossibility theorems, which starts from the Hansson–Denicolò theorem. By contrast, Yu (2013) avoids the extra step of establishing the Muller–Satterthwaite theorem, and derives them from Theorem 4, which reveals the fundamental difficulty to be pairwise consistency requirements. But the definition of (CM) involves one more attributive than (IIA) does. As a result, Yu (2013) demands a higher level of mathematical maturity. When we compare these two approaches for teaching purposes, the tradeoff between succinctness and ease thus depends on the audience.

First, Arrow's impossibility theorem (stated in Sect. 6) is a straightforward consequence of the Hansson–Denicolò theorem as shown by Denicolò (1993).

Proof of Arrow's theorem A SCF F^R is *derived from* R in that for any $\vec{\succ}$, $a_i R(\vec{\succ}) a_j$ for every $j \neq i$ is necessary and sufficient for $F^R(\vec{\succ}) = a_i$, i.e., it selects the alternative ranked highest by R. F^R obviously satisfies (WP) and (IIA), so the Hansson-Denicolò theorem says that F^R presents a social choice dictator n. She is a social preference dictator too. If $a_i \succ_n a_j$, individual n rules $F^R_{ij}(\vec{\succ}) = a_i$, so $a_i R(\vec{\succ}) a_j$ by (SP-AIIA). □

Based on the Hansson–Denicolò theorem, we prove the Muller–Satterthwaite theorem (stated in Sect. 5) directly, which is a popular pathway to the theorem of Gibbard (1973) and Satterthwaite (1975).[19] Denicolò (1993) does not offer this development.

Proof of Muller–Satterthwaite: To show (IIA), let F satisfy (M) and $\vec{\succ}$ and $\vec{\succ}'$ agree on $\{a_i, a_j\}$. Suppose $F(\vec{\succ}) = a_i$ and $F(\vec{\succ}') = a_j$. By (M), $F_{ij}(\vec{\succ})$ should be both a_i and a_j, a contradiction.

Suppose that (WP) is violated, i.e., for some profile $\vec{\succ}$, $F(\vec{\succ}) = a_j$, where a_i dominating a_j. By (M), $F_{ij}(\vec{\succ}) = a_j$. By (O), there exists $\vec{\succ}'$ such that $F(\vec{\succ}) = a_i$. By (M), $F_{ij}(\vec{\succ}) = a_i$, a contradiction. □

When we replace the preferences of individual n in $\vec{\succ}$ with $\succ'_n \in \mathcal{P}$, denote the new profile $(\succ'_n, \vec{\succ}_{-n})$. *Strategy-proofness* ensures truth-telling by always switching to a worse alternative for an individual if only her misreport alters the outcome.

Definition 25 (*SP*). A SCF F is *strategy-proof* if $F(\succ'_n, \vec{\succ}_{-n}) \neq F(\vec{\succ})$ implies $F(\vec{\succ}) \succ_n F(\succ'_n, \vec{\succ}_{-n})$ for every n, every $\vec{\succ}$, and every $\succ'_n$.

Theorem 11 (Gibbard–Satterthwaite). *If F satisfies (O) and (SP), then it is dictatorial.*

Proof It is a consequence of Theorem 5 and the sufficiency part of Lemma 10. □

[19] See Mas-Colell, Whinston, and Green (1995, p. 874) and Reny (2001). The theorem is a pathbreaking development that brought strategic considerations into the field, the importance of which needs no further elaboration.

Muller and Satterthwaite (1977) demonstrate the equivalence of (SP) and (M). Our proof of the claim that (M) implies (SP) is shorter than the original version.

Lemma 10 *F satisfies (M) if and only if it satisfies (SP).*[20]

Proof Sufficiency: Let $F(\vec{\succ}) = a_i$ and $\vec{\succ}'$ maintain a_i's positions in $\vec{\succ}$. Define $\vec{\succ}'' \equiv (\succ_1', \vec{\succ}_{-1})$, then $\vec{\succ} = (\succ_1, \vec{\succ}''_{-1})$. Suppose $a_j = F(\vec{\succ}'') \neq a_i$. (SP) demands $a_i \succ_1 a_j$ and $a_j \succ_1' a_i$, so $\vec{\succ}'$ lowers a_i's positions relative to a_j for individual 1, contradicting the assumption. So $F(\vec{\succ}'') = a_i$. Following the same logic, we can update $\vec{\succ}$ to $\vec{\succ}'$ one by one without altering the choice.

Necessity: Suppose $a_i = F(\succ_n', \vec{\succ}_{-n}) \succ_n F(\vec{\succ}) = a_j$. Consider $\succ_n''$ with $a_i \succ_n'' a_j \succ_n'' a_k$ for every $k \notin \{i, j\}$. We have $(\succ_n'', \vec{\succ}_{-n})$ maintains a_i's position in $(\succ_n', \vec{\succ}_{-n})$ and a_j's position in $\vec{\succ}$, but $F(\succ_n'', \vec{\succ}_{-n})$ cannot be both a_i and a_j. □

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